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Universal scaling of critical quasiperiodic orbits in a class of twist maps

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Abstract. Recently we have shown that the fractal properties of the critical invariant circles of the standard map, as summarized by the $f(\alpha)$ spectrum and the generalized dimensions $D(q)$, depend only on the tails in the continued fraction expansion of the corresponding rotation numbers in (Burić N, Mudrinić M and Todorović K 1997 *J. Phys. A: Math. Gen.* **30** L161). In the present paper this result is extended on the whole class of sufficiently smooth area-preserving twist maps of cylinders. We present numerical evidence that the $f(\alpha)$ and $D(q)$ are the same for all critical invariant circles of any such map which have the rotation numbers with the same tail.

1. Introduction

This paper is devoted to a study of universal properties of invariant sets of Hamiltonian dynamical systems with two degrees of freedom, which are important in many areas of physics. Such systems usually have very complicated behaviour described typically by various fractal functions. On the other hand, the scaling properties of these fractal structures seem to be, to a certain extent, independent of the details of the particular system and universal over classes of equivalent types of orbits. The scaling of these apparently self-similar structures is usually described by various fractal dimensions and scaling functions, which can be approximately calculated using methods of the renormalization theory [2]. Sometimes the scaling properties can be used as a computational tool for efficient approximate calculation of these fractal objects [5]. Despite the obvious importance, it is still not known which classes of systems and corresponding fractal objects have the same scaling properties.

Hamiltonian systems with two degrees of freedom are conveniently studied using area-preserving maps, which come in as the corresponding Poincaré maps of the Hamiltonian system. In this paper we shall consider typical area-preserving twist maps of a cylinder [3], given by the following equations:

$$T : \begin{cases} p_{t+1} = p_t - \dfrac{k}{2\pi} g(2\pi x_t) \\ x_{t+1} = (x_t + p_{t+1}) \bmod 1 \qquad x_t \in \mathbb{S}^1,\ p_t \in \mathbb{R} \end{cases} \tag{1}$$

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where $g(x)$ is any sufficiently smooth periodic function and k is a parameter. These maps possess different types of regular orbits, such as: periodic, quasiperiodic and homoclinic. Besides the regular orbits there are usually many chaotic orbits, which can go over large parts of the phase space.

The set of quasiperiodic orbits of maps given by formula (1) are parametrized by irrational values of the rotation number ν, which is defined as follows

$$\nu := \lim_{i\to\infty} \frac{\bar{\mathbb{T}}^i \bar{x} - \bar{x}}{i} \qquad \bar{x} \in \mathbb{R} \tag{2}$$

where $\bar{\mathbb{T}}$ is the lift of the map (1).

The closure of a typical quasiperiodic orbit with a given ν, for a sufficiently small value of the perturbation parameter k, is an invariant curve diffeomorphic to the circle. For a large value of the parameter the closure of the quasiperiodic orbit with the same ν is a more complicated invariant set, called cantorus. It is an invariant Cantor set imbedded in the phase space. There is at least one value of the perturbation parameter $k = K(\nu)$, which depends on the rotation number and is called critical, such that the quasiperiodic orbit is still dense on the circle, but the invariant measure given by the orbit is singular with respect to the Lebesgue measure on the circle. Points of the critical orbit, although dense on the circle, are distributed in an inhomogeneous way, described by a fractal density function with a nontrivial self-similar structure.

It is generally believed that the distribution of points on the critical invariant circles is in some sense universal for a large class of area-preserving maps. However, it is still not clear in what sense the critical orbits are universal and what the relevant classes of equivalence are.

It is clear that the self-similar structure generated by the critical orbit should depend somehow on the rotation number of the orbit. Furthermore, each periodic and quasiperiodic orbit of the map (1) generate a corresponding map of the circle, and it is known that in the case of maps of the circle different degrees of inflection of the maps imply different fractal properties of the orbits [6]. Thus another relevant quantity in the case of maps (1) might be the degree of inflection points of the functions $g(x)$.

The idea that the qualitative behaviour of the critical orbits should depend only on the tail of the continued fraction expansion (CFE) of the corresponding rotation number is central in the renormalization theory for arbitrary rotation numbers [2]. Also, results of the method of modular smoothing [5] indicate that the dominant singularities of the critical circles of a single map are of the same type for circles with the rotation numbers related by some modular transformation. Such rotation numbers have the same tail in their CFE. In [1] we used this as the guiding idea to show that the fractal properties of critical quasiperiodic orbits for a single map (namely the standard map where $g(x) = \sin(2\pi x)$) depend only on the tail of the CFE. It is the purpose of this paper to show that the dependence of the fractal properties of the critical orbits on their rotation numbers is universal for a class of smooth area-preserving maps.

The global self-similar structure of fractals with nontrivial scaling is usually described by the spectrum of singularities $f(\alpha)$, related to the spectrum of generalized (Renyi) dimensions [4] (the definitions are briefly recapitulated later in the text). Our numerical calculations, presented in [1], show that the singularity spectra $f(\alpha)$ and the spectra of fractal dimensions $D(q)$ of the invariant measures $\mu(\nu)$ are equal for all orbits of the standard map with the rotation numbers with the same tails in the CFE, and are different if the tails of the CFE are different. Thus, the critical circles of the standard map can be divided into equivalence classes with respect to their fractal properties. Members of the same class have the same

$f(\alpha)$, $D(q)$ and the tail in the CFE and different tails imply different $f(\alpha)$ and $D(q)$. In this paper we shall present numerical evidence that the functions $f(\alpha)$ and $D(q)$ are equal for all critical circles of periodic area-preserving maps of the form (1), where $g(x)$ is any finite trigonometric polynomial, provided that the corresponding rotation numbers have the same tail in the CFE. Thus the fractal properties of critical curves do not depend either on the initial part of the CFE of the rotation number, or on the details of the map, provided that the map is of the form (1) with $g(x)$ a finite trigonometric polynomial. Furthermore, based on the results of [7] (to be reviewed in a later section), we conjecture that the degree of smoothness of the function $g(x)$ which is sufficient for our results is the same as the one in the KAM theory for such maps [10].

The fractal properties of the critical Torii were described by $f(\alpha)$ and $D(q)$ for the first time in [8]. However Osbaldestin and Sarkis calculated $f(\alpha)$ and $D(q)$ for a few critical circles without systematic exploration and discussion of the dependence on the number-theoretical properties of the rotation numbers. The result that the information dimension D_1 is the same for all orbits with the same tail in the CFE for a class of maps is implicitly contained in the work of Hunt *et al* [9]. Our results show that not only D_1 but the whole spectrum $D(q)$ does not depend either on the map (provided it is arbitrary periodic and sufficiently smooth), or on the details of the rotation number, but depends only on the tail of the rotation number. This extends the results of [9] where only the information dimension D_1 was considered and no systematic exploration of the class of maps was given.

This paper is organized as follows. In section 2 we recapitulate the general definition of the function $f(\alpha)$ and its relation to the spectra of fractal dimensions $D(q)$. Then we discuss details of the application of the general construction in the case of critical curves of maps (1). Section 3 contains a description of our calculations and our main results, which are summarized and discussed in section 4.

2. Description of the global scaling of the critical orbits

The function $f(\alpha)$ contains complete information about the global scaling of complicated fractals imbedded in one-dimensional sets. We shall now briefly state the steps in the procedure which give a well-defined $f(\alpha)$ function for the critical orbits of maps given by (1). The procedure is essentially based on the same ideas as in Green's procedure for calculations of the critical values $K(\nu)$ [11]. The relation between properties of quasiperiodic orbits and of the corresponding sequences of periodic orbits is fully exploited in the variational approach to the theory of twist maps by Aubry [12]. The main assumption is that the sequence of periodic orbits, with rotation numbers given as the continued fraction convergents of the irrational rotation number of the considered quasiperiodic orbit, asymptotically approximates the quasiperiodic orbit even at the critical values of the parameter. We were motivated by the same ideas in our choice of the partitions and measures needed for the calculations of $f(\alpha)$ and $D(q)$.

As pointed out before the critical orbit can still be considered as an orbit of a homeomorphism of the circle. To define the function $f(\alpha)$ one considers an infinite set of partitions of the circle. The nth partition contains Q_n pieces (Q_n is increasing with n) labelled by an index i: $1 \leqslant i \leqslant Q_n$. The size of the ith piece is denoted by l_i, and the probability that a point of the orbit is in the ith piece is denoted by p_i. One then assumes that p_i scales as $p_i \approx l_i^{\alpha}$, and defines the function $f(\alpha)$ as the Hausdorf dimension of the set of points having exponent α. If the partitions and the measure are appropriate for the considered self-similar structure then the function $f(\alpha)$ can be calculated from the properties of the limit of the sequence of partitions.

To compute $f(\alpha)$ of the critical circle with an arbitrary rotation number ν we follow the procedure used in [7] for computation of $f(\alpha)$ for the critical circle with the rotation number equal to the golden mean $\nu = [0, 1^{\infty}]$. Similar, but not the same, partitions which gave the same asymptotic results were used in [8, 9]. In our computations the partitions of the circle are given by the points of the periodic orbits which approximate the critical quasiperiodic orbit. The rotation numbers of the set of periodic orbits are chosen as the successive continued fraction convergents of the irrational $\nu = [0, a_1, a_2 \ldots]$. Thus the nth partition of the circle is given by Q_n points of the periodic orbit with the rotation number $P_n/Q_n = \{0, a_1, a_2, \ldots, a_n\}$ at the parameter value $k = K(\nu)$. There are two such orbits, one elliptic and one hyperbolic, but our results for $f(\alpha)$ and $D(q)$ did not depend on which of the two types of orbits were used to generate the partitions. If (p_i, x_i) and (p_{i+1}, x_{i+1}) are coordinates of the two neighbouring points on the periodic orbit then

$$l_i(n) = [(p_i - p_{i+1})^2 + (x_i - x_{i+1})^2]^{1/2}. \tag{3}$$

The measure of $l_i(n)$ is defined as $p_i(n) = 1/Q_n$, and the partition function of the partition with Q_n points is then given by

$$\Gamma_n(q_n, \tau_n) = \sum_{i=1}^{Q_n} \frac{p_i^{q_n}}{l_i^{\tau_n}} = n^{-q_n} \sum_{i=1}^{Q_n} l_i^{-\tau_n}. \tag{4}$$

The partition function is of the order unity when

$$\tau_n = (q_n - 1) D_n(q_n) \tag{5}$$

and

$$D(q) = \lim_{n\to\infty} D_n(q_n) \tag{6}$$

where $D(q)$ is the set of generalized dimensions of the fractal. $f(\alpha)$ is then given by

$$f_n(\alpha_n) = q_n \alpha_n(\tau_n) - \tau_n \qquad \alpha_n(\tau_n) = \frac{\mathrm{d}\tau_n}{\mathrm{d}q_n} \tag{7}$$

and

$$f(\alpha) = \lim_{n\to\infty} f_n(\alpha_n). \tag{8}$$

In summary the procedure to calculate $f(\alpha)$ and $D(q)$ for a critical quasiperiodic orbit consists of the following steps. Estimate the critical value $K(\nu)$ using, for example, the Green criterion [11], and then calculate $\tau_n(q_n)$ for the partitions generated by the periodic orbits with the rotation numbers given by the successive continued fraction convergents. Each $\tau_n(q_n)$ gives the corresponding α_n, $f_n(\alpha_n)$ and $D_n(q_n)$. Finally estimate the limit of $f_n(\alpha_n)$ and $D_n(q_n)$. The appropriate choice of the partitions and the measure is manifested in a relatively fast convergence of the results, and makes the actual computations feasible. The convergence can be enhanced by the usual ratio trick, and the condition that $\Gamma(\tau_n, q_n)/\Gamma(\tau_{n+1}, q_{n+1}) = 1$ can be used to provide explicit formulae for q_n and α_n as functions of τ_n [7]. The formulae are as follows:

$$q_n = \frac{1}{\ln(Q_{n-1}/Q_n)} \left[\ln\left(\sum_{i=1}^{Q_{n-1}} l_i^{-\tau}(n-1) \right) - \ln\left(\sum_{i=1}^{Q_n} l_i^{-\tau}(n) \right) \right] \tag{9}$$

$$\alpha_n = \ln\left(\frac{Q_{n-1}}{Q_n}\right) \left[\frac{\sum_{i=1}^{Q_n} l_i^{-\tau}(n) \ln[l_i(n)]}{\sum_{i=1}^{Q_n} l_i^{-\tau}(n)} - \frac{\sum_{i=1}^{Q_{n-1}} l_i^{-\tau}(n-1) \ln[l_i(n-1)]}{\sum_{i=1}^{Q_{n-1}} l_i^{-\tau}(n-1)} \right]^{-1} \tag{10}$$

where Q_n is the denominator of the nth continued fraction convergent P_n/Q_n of the irrational rotation number ν.

3. Calculations and results

Our conclusions are based on extensive calculations of the function $f(\alpha)$ for various quasiperiodic orbits of various maps of the form given by (1). In all calculations the functions $g(x)$ were trigonometric polynomials with finite number of terms:

$$g(x) = \sum_{i=1}^{i=N} a_i \sin(\mathrm{i}2\pi x) + b_i \cos(\mathrm{i}2\pi x). \tag{11}$$

Any real function of the form (11) is smooth, with derivatives that can have either finite or zero values, so that the degrees of the possible inflection points are always bigger than or equal to 3. We have considered several such functions including examples of purely even or odd functions, functions with various degrees of inflection and with various numbers of terms.

Periodicity of the functions (11) is 2π. Maps with functions of the same type but with arbitrary periods of the following form

$$g(x) = \sum_{i=1}^{i=N} a_i \sin\left(\mathrm{i}\frac{2\pi}{L}x\right) + b_i \cos\left(\mathrm{i}\frac{2\pi}{L}x\right) \qquad L \in Z^+ \tag{12}$$

are mapped onto the 2π periodic maps by the following changes of the coordinates:

$$x \to Lx \qquad p \to Lp.$$

These transformations induce modular transformations of the rotation numbers, which do not change the tails of the CFE. Since our results depend only on the tail of the CFE we see that it is enough to consider only 2π periodic functions.

For each of the considered maps we calculated $f(\alpha)$ of the critical quasiperiodic orbits with the rotation numbers of the form:

$$\nu = \{0, a_1, a_2, \ldots, a_n, j^\infty\}$$

where $a_1, a_2, \ldots, a_n$ denote the first n coefficients in the CFE which are variable and j^∞ denotes an infinite sequence of equal integers j, namely the tail of the CFE. There are many Diophantine irrationals which are not of the form (3), such that the corresponding quasiperiodic orbits behave similarly to those with rotation numbers of the form (3) (KAM theory, break-up into the Cantori etc). We did not treat such orbits explicitly, due to numerical limitations, but we do not see any reason why the conclusions based on considered orbits should not be valid for any set of orbits with Diophantine rotation numbers. Furthermore, numerical problems (to be discussed later) limited us to the set of only a few (five or so) orbits with the same tail. But as we shall see, the numerical evidence is so convincing that we have no reason to suspect that other orbits, which we could not calculate, will suddenly start to behave differently.

Our main result is illustrated in figures 1(a) and (b). The figures represent $f(\alpha)$ and $D(q)$ for the critical quasiperiodic orbits with specially selected rotation numbers, and for several maps of form (1). In the previous paper [1] we demonstrated that in the case of the standard map $f(\alpha)$ and $D(q)$ are the same for all rotation numbers with the same tail in the CFE. Figures 1(a) and (b) show that this is also true for the map with three Fourier terms. The full curve illustrates $f(\alpha)$ and $D(q)$ for a class of quasiperiodic orbits with the same tail in the CFE as the tail of the golden mean, that is $[1^\infty]$. The curves for quasiperiodic orbits with different rotation numbers in this class are indistinguishable, so that there is nothing but the one curve to plot. The same result is true for the classes given by other tails. The broken curve represents $f(\alpha)$ and $D(q)$ for the class with the tail equal to $[2^\infty]$

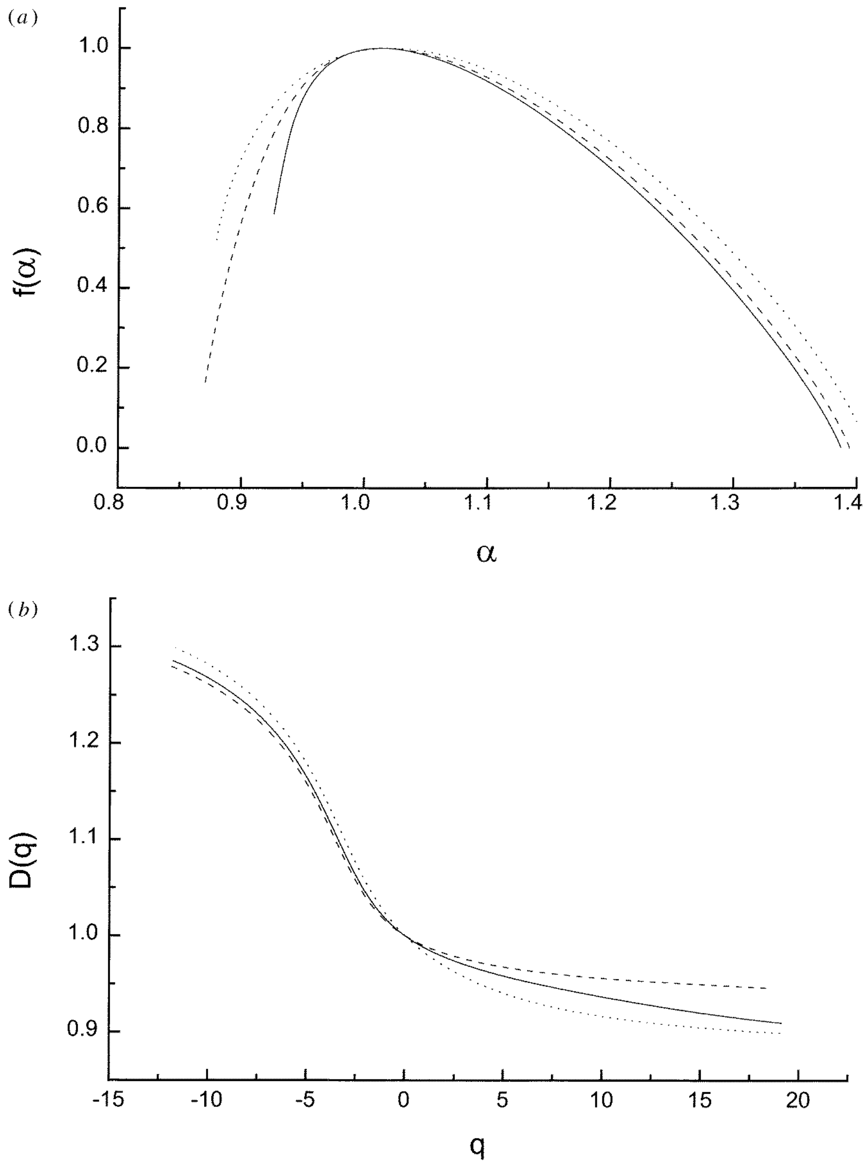

Figure 1. (a) $f(\alpha)$ and (b) $D(q)$ curves for the critical circles of a few typical maps in several equivalence classes. Shown are the classes corresponding to the tails equal to $[1^{\infty}]$ (full), $[2^{\infty}]$ (broken), and $[3^{\infty}]$ (dotted).

and the dotted curve represents the class with the tail equal to $[3^{\infty}]$. The same result was obtained for other considered orbits and/or maps. We systematically explored four typical maps of the form (1). The maps are chosen to represent various typical cases: one is even with three Fourier terms, one is with the inflection point at zero of order five, one is with the inflection point different from zero and one is the standard map. Longer periodic orbits give better accuracy, and $f(\alpha)$ and $D(q)$ for orbits with the same tail cannot be visually distinguished. The accuracy of the calculations of the critical values $K(\nu)$ needed to obtain visually the same $f(\alpha)$ was greater than or equal to four significant digits. The curves for different maps with the same tails as those examined here cannot be distinguished from those presented in the figures. The calculations for maps with more Fourier terms are consistent but incomplete because the numerical calculations of the critical periodic orbits become much more difficult.

Combined with the previous results this gives our main conclusion: the functions $f(\alpha)$ and $D(q)$ for different classes of the critical quasiperiodic orbits are different, but within the class (given by the tail of the CFE of the rotation number) these functions are the same for all maps of the form (1), where $g(x)$ is any finite trigonometric polynomial. $f(\alpha)$ and $D(q)$ do not depend either on the map or on the details of the rotation number but only on the tail in the CFE of the rotation number.

Our calculations of $f(\alpha)$ and $D(q)$ required knowledge of quite long periodic orbits at the critical values of the parameter. These orbits are used to determine the values of $K(\nu)$ and to calculate the partition functions. The final results, $f(\alpha)$ and $D(q)$, are extremely sensitive to the value of the parameter k. In order to establish our conclusions we needed to calculate $K(\nu)$ to at least five significant digits, which implies calculations of very long periodic orbits (periods of about a few thousand). If one of the initial coefficients in the CFE of the rotation number is large, then the long periodic orbit, for the near-critical value of the perturbation parameter, could be easily mistaken for a shorter orbit in its neighbourhood which is stable for such perturbation. Calculation of such orbits is obviously a very difficult numerical problem. Usually the methods of calculations are based on some form of the Newton method, and are relatively easy only in the case of orbits with small periods. One of the methods of this type, specially designed to treat long near-critical orbits, is the one developed some time ago in [13]. Recently, Vrahatis [14] proposed a procedure based on a generalized bisection method. Both of these methods have been applied with similar success. In any case the calculations of long periodic orbits close to the bifurcation points are far from being straightforward, and the efficiency of the methods depends on the special techniques to determine a good initial guess for the Newton method, or a good choice of the initial characteristic polihedra. The computational accuracy has limited us to a relatively small set of about five quasiperiodic orbits in each considered class and to the maps with only a few terms. However, we believe that the results are typical, and as such support our conclusions.

4. Summary and discussion

Precise characterization of the universal properties of phase portraits is a long-standing problem in the theory of Hamiltonian dynamical systems. In this paper we have presented our results about universality of the fractal properties of the critical orbits. Our numerical evidence indicates that: $f(\alpha)$ and $D(q)$, for the critical orbits of the area-preserving twist maps given by finite trigonometric polynomials, depend only on the tail of the CFE of the irrational rotation number. $f(\alpha)$ and $D(q)$ are equal for all critical orbits and for all maps in the class, if the tails in the CFE of the rotation numbers of the orbits are equal. If the tails are different $F(\alpha)$ and $D(q)$ are different.

Our results are obtained by numerical computation of carefully selected examples of the maps and the rotation numbers from the considered class. Although we cannot prove the results, we believe that the typical cases have been tested and that the numerical evidence is convincing.

We have considered the class of maps (1) given by the finite trigonometric polynomials. Hu and Shi [7] calculated $f(\alpha)$ for the critical orbits, with the rotation number equal to the inverse golden mean, of maps in a family given by the periodic extension of the functions:

$$g(x) = x(1 - |2x|^{z-1}) \tag{13}$$

where z is a real positive parameter. When z is an integer the maps are analytic with the inflection point at $x = 0$ of the degree z. When z is not an integer the map is of a finite smoothness. Hu and Shi concluded that if $z \geqslant 3$, so that the map is C^k with $k > 3$, then $f(\alpha)$, for the golden curve, does not depend on z. Furthermore Hu and Shi observed and studied the disappearance and reappearance of the smooth golden curve as the parameter is monotonically increased. We considered invariant curves of the maps (13) with other rotation numbers and obtained the same conclusion. Also, $f(\alpha)$ for the critical orbits of the maps with $z \geqslant 3$ and all considered rotation numbers are equal and the same as for the maps given by the trigonometric polynomials. We did not study the reappearance of the smooth invariant curves.

Based on our calculations and on the work of Hu and Shi we conjecture that $f(\alpha)$ and $D(q)$ are equal for all critical orbits with equal tails in the CFE of the rotation numbers and for all twist maps of cylinders, provided that the maps and the rotation numbers satisfy the conditions of the KAM theory for the twist maps [10].

It would be interesting and useful, although difficult from the theoretical and computational points of views, to extend our results onto the simplectic twist maps in more dimensions.

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